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n-Dimensional Extended Index Matrices Part 2

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Abstract. Index Matrices (IMs) are extensions of the ordinary matrices. They are also object of extensions and modifications, e.g., extended index matrices. The present paper is the second part of our research over n-dimensional extended IMs. Here, we introduce new operations and relations over these matrices.
Keywords: Function, Index matrix, Operation, Relation.

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1 Introduction

The concept of n-Dimensional Extended Index Matrix (n-DEIM), for arbitrary natural number $n \geq 2$, was introduced for the first time in [13] as extension of the standard and of the extended index matrices (see, e.g., [1, 2, 3, 5, 8, 9, 20]).

Here, we will use the notations from [13] without discussions. In this paper, we introduced the definition of the n-DEIM, the operations "addition", "termwise multiplication", "multiplication", "structural subtraction", "multiplication with a constant", "projection", "reduction", "substitution" (of the IM-indices), and six relations.

Here, we introduce a new operation "substitution" (of the IM-elements), hierarchical operators, aggregation operations, and six relations for the case, when the n-DEIM are propositions or predicates. Also, we introduce n-DEIMs with function-type of elements.

2 Operations "substitution" of the IM-elements

Let the n-DEIM A be given. Following [13], we mention that the n-DEIM with index sets $K_1, K_2, ..., K_n$ ($K_1, K_2, ..., K_n \subseteq \mathcal{I}^*$) and elements from set $\mathcal{X}$ is the object:

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, ..., k_{i,m_i}\}$, $m_i \geq 1$ and $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{X}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$.

In [13], the first operations "substitution" are introduced.

Let for brevity $1 \leq i < j \leq n$. Then the local substitution of a value of A is defined by

$$\left[\left\langle i, j; k_{i,p_i}, k_{j,q_j}; \frac{b_{k_{i,p_i},k_{j,q_j}}}{a_{k_{i,p_i},l_{j,q_j}}}\right\rangle\right]$$

$$\left\{ \begin{array}{c|ccc} \begin{array}{c}\langle k_{1,s_1}, ..., k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, \\ k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n}\rangle\end{array} & \dots & k_{j,q_j 2} & \dots \\ \hline \vdots & \vdots & \vdots & \vdots \\ k_{i,p_i} & \dots & a_{k_{i,p_i},k_{j,q_j}} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right.$$

$$|\langle k_{1,s_1}, ..., k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n}\rangle$$
$$\in K_1 \times \dots K_{i-1} \times K_{i+1} \times \dots \times K_{j-1} \times K_{j+1} \times K_n\}$$

$$= \left\{ \begin{array}{c|ccc} \begin{array}{c}\langle k_{1,s_1}, ..., k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, \\ k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n}\rangle\end{array} & \dots & k_{j,q_j 2} & \dots \\ \hline \vdots & \vdots & \vdots & \vdots \\ k_{i,p_i} & \dots & b_{k_{i,p_i},k_{j,q_j}} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right.$$

$$|\langle k_{1,s_1}, ..., k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n}\rangle$$
$$\in K_1 \times \dots K_{i-1} \times K_{i+1} \times \dots \times K_{j-1} \times K_{j+1} \times K_n\}.$$

3 Hierarchical operators over EIMs

In [4, 8], two hierarchical operators are defined over 2-dimensional IM. They are applicable to EIM, when their elements are not only numbers, variables, etc, but also whole (new) IMs. First, we give their definitions, after this a new form of a hierarchical operator over EIMs and finally, its definition when it is defined over an n-dimensional IM.

Let A be a 2-dimensional EIM and let its element a_{k_f,e_g} be an IM by itself:

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$K \cap P = L \cap Q = \emptyset.$$

The first hierarchical operator is

$$A|(a_{k_f,l_g}) = [(K - \{k_f\}) \cup P, (L - \{l_g\}) \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K - \{k_f\} \text{ and } v_w = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}.$$

Let us assume that in the case when a_{k_f,l_g} is not an element of IM A, then

$$A|(a_{k_f,l_g}) = A.$$

From the first definition of a hierarchical operator it follows that

$$A|(a_{k_f,l_g})$$

$$= \begin{array}{c|cccccccccc} & l_1 & \ldots & l_{g-1} & q_1 & \ldots & q_u & l_{g+1} & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_{g-1}} & 0 & \ldots & 0 & a_{k_1,l_{g+1}} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{f-1} & a_{k_{f-1},l_1} & \ldots & a_{k_{f-1},l_{g-1}} & 0 & \ldots & 0 & a_{k_{f-1},l_{g+1}} & \ldots & a_{k_{f-1},l_n} \\ p_1 & 0 & \ldots & 0 & b_{p_1,q_1} & \ldots & b_{p_1,q_v} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_u & 0 & \ldots & 0 & b_{p_u,q_1} & \ldots & b_{p_u,q_v} & 0 & \ldots & 0 \\ k_{f+1} & a_{k_{f+1},l_1} & \ldots & a_{k_{f+1},l_{g-1}} & 0 & \ldots & 0 & a_{k_{f+1},l_{g+1}} & \ldots & a_{k_{f+1},l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_{g-1}} & 0 & \ldots & 0 & a_{k_m,l_{g+1}} & \ldots & a_{k_m,l_n} \end{array}.$$

From this form of the IM $A|(a_{k_f,l_g})$ we see that for the hierarchical operator the following equality holds.

Theorem 1 (see [8]). Let

$$A = [K, L, \{a_{k_i,l_j}\}]$$

be an IM and let

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}]$$

be its element. Then

$$A|(a_{k_f,l_g}) = (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f,l_g}.$$

In [8], the first hierarchical operator is modified so that all the information from the IMs, participating in it, be preserved. The new – second – form of this operator for the above defined IM A and its fixed element a_{k_f,l_g}, is

$$A|^*(a_{k_f,l_g})$$

$$
= \begin{array}{c|ccccccccc}
 & l_1 & \dots & l_{g-1} & q_1 & \dots & q_u & l_{g+1} & \dots & l_n \\
\hline
k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_{g-1}} & a_{k_1,l_g} & \dots & a_{k_1,l_g} & a_{k_1,l_{g+1}} & \dots & a_{k_1,l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k_{f-1} & a_{k_{f-1},l_1} & \dots & a_{k_{f-1},l_{g-1}} & a_{k_{f-1},l_g} & \dots & a_{k_{f-1},l_g} & a_{k_{f-1},l_{g+1}} & \dots & a_{k_{f-1},l_n} \\
p_1 & a_{k_f,l_1} & \dots & a_{k_f,l_{g-1}} & b_{p_1,q_1} & \dots & b_{p_1,q_v} & a_{k_f,l_{g+1}} & \dots & a_{k_f,l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_u & a_{k_f,l_1} & \dots & a_{k_f,l_{g-1}} & b_{p_u,q_1} & \dots & b_{p_u,q_v} & a_{k_f,l_{g+1}} & \dots & a_{k_f,l_n} \\
k_{f+1} & a_{k_{f+1},l_1} & \dots & a_{k_{f+1},l_{g-1}} & a_{k_{f+1},l_g} & \dots & a_{k_{f+1},l_g} & a_{k_{f+1},l_{g+1}} & \dots & a_{k_{f+1},l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k_m & a_{k_m,l_1} & \dots & a_{k_m,l_{g-1}} & a_{k_m,l_g} & \dots & a_{k_m,l_g} & a_{k_m,l_{g+1}} & \dots & a_{k_m,l_n}
\end{array} .
$$

Now, the following assertion is valid.

Theorem 2 (see [8]). Let

$$A = [K, L, \{a_{k_i,l_j}\}]$$

be an IM and let

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}]$$

be its element. Then

$$A|^*(a_{k_f,l_g})$$

$$= (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f,l_g} \oplus [P, L - \{l_g\}, \{c_{x,l_j}\}] \oplus [K - \{k_f\}, Q, \{d_{k_i,y}\}],$$

where for each $x \in P$ and for each $l_j \in L - \{l_g\}$,

$$c_{x,l_j} = a_{k_f,l_j}$$

and for each $k_i \in K - \{k_f\}$ and for each $y \in Q$,

$$d_{k_i,y} = a_{k_i,l_g}.$$

In [8], other representations of IMs $A|(a_{k_f,l_g})$ and $A|^*(a_{k_f,l_g})$ are given, using other operations defined over IMs.

Here, following [12], we discuss a 2-dimensional EIM each element of which is an EIM and an operator that modify this EIM to a standard EIM.

Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an EIM, where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{X}$. Let each its element a_{k_f,e_g} be an EIM by itself:

$$a_{k_f,l_g} = [P_{k_f,l_g}, Q_{k_f,l_g}, \{b_{k_f,l_g,p_u,q_v}\}],$$

where $1 \leq f \leq m$, $1 \leq g \leq n$, $1 \leq u \leq r_{f,g}$, $1 \leq v \leq s_{f,g}$ and

$$P_{k_f,l_g} = \{p_{k_f,l_g,1}, ..., p_{k_f,l_g,r_{f,g}}\},$$

$$Q_{k_f,l_g} = \{q_{k_f,l_g,1}, ..., q_{k_f,l_g,s_{f,g}}\},$$

$$K \cap P_{k_f,l_g} = L \cap Q_{k_f,l_g} = \emptyset \qquad (1)$$

and for every four indices $k_f, k_h \in \mathcal{I}^*$ and $l_g, l_i \in \mathcal{I}^*$:

$$P_{k_f,l_g} \cap P_{k_h,l_i} = Q_{k_f,l_g} \cap Q_{k_h,l_i} = \emptyset. \qquad (2)$$

The new (third) hierarchical operator is defined by

$$A|^*$$

$$= \begin{array}{l|cccccccc}
 & q_{k_1,l_1,1} & \dots & q_{k_1,l_1,s_{1,1}} & \dots & q_{k_m,l_n,1} & \dots & q_{k_m,l_n,s_{m,n}} \\ \hline
p_{k_1,l_1,1} & a_{k_1,l_1,1,1} & \dots & a_{k_1,l_1,1,s_{1,1}} & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_1,l_1,r_{1,1}} & a_{k_1,l_1,r_{1,1},1} & \dots & a_{k_1,l_1,r_{1,1},s_{1,1}} & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_m,l_n,1} & 0 & \dots & 0 & \dots & a_{k_m,l_n,1,1} & \dots & a_{k_m,l_n,1,s_{m,n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_m,l_n,r_{m,n}} & 0 & \dots & 0 & \dots & a_{k_m,l_n,r_{m,n},1} & \dots & a_{k_m,l_n,r_{m,n},s_{m,n}}
\end{array}.$$

We see that conditions (1) and (2) are important for the correctness of the definition.

For example, if

$$A = \begin{array}{c|cc}
 & l_1 & l_2 \\ \hline
k_1 & a_{1,1} & a_{1,2} \\
k_2 & a_{2,1} & a_{2,2} \\
k_3 & a_{3,1} & a_{3,2}
\end{array},$$

where

$$a_{1,1} = \begin{array}{c|cc}
 & q_{1,1} & q_{1,2} \\ \hline
p_{1,1} & b_{1,1} & b_{1,2} \\
p_{1,2} & b_{2,1} & b_{2,2} \\
p_{1,3} & b_{3,1} & b_{3,2}
\end{array},$$

$$a_{1,2} = \begin{array}{c|ccc}
 & q_{2,1} & q_{2,2} & q_{2,3} \\ \hline
p_{2,1} & c_{1,1} & c_{1,2} & c_{1,3} \\
p_{2,2} & c_{2,1} & c_{2,2} & c_{2,3}
\end{array},$$

$$a_{2,1} = \begin{array}{c|c}
 & q_{3,1} \\ \hline
p_{3,1} & d_{1,1}
\end{array},$$

$$a_{2,2} = \begin{array}{c|cc}
 & q_{4,1} & q_{4,2} \\ \hline
p_{4,1} & e_{1,1} & e_{1,2} \\
p_{4,2} & e_{2,1} & e_{2,2} \\
p_{4,3} & e_{3,1} & e_{3,2}
\end{array},$$

$$a_{3,1} = \begin{array}{c|c}
 & q_{5,1} \\ \hline
p_{5,1} & f_{1,1} \\
p_{5,2} & f_{2,1}
\end{array},$$

and

$$a_{3,2} = \begin{array}{c|cc}
 & q_{6,1} & q_{6,2} \\ \hline
p_{6,1} & g_{1,1} & g_{1,2}
\end{array},$$

then

$$A|^* = \begin{array}{c|ccccccccccc}
 & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,1} & q_{4,2} & q_{5,1} & q_{6,1} & q_{6,2} \\ \hline
p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{1,2} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{2,1} & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{3,1} & 0 & 0 & 0 & 0 & 0 & d_{1,1} & 0 & 0 & 0 & 0 & 0 \\
p_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & 0 & 0 & 0 \\
p_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & 0 & 0 \\
p_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} & 0 & 0 & 0 \\
p_{5,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{1,1} & 0 & 0 \\
p_{5,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{2,1} & 0 & 0 \\
p_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,1} & g_{1,2}
\end{array}.$$

Now, keeping condition (1), we can change condition (2) with a weaker one: for every four indices $k_f, k_h \in \mathcal{I}^*$ and $l_g, l_i \in \mathcal{I}^*$:

$$P_{k_f,l_g} \times Q_{k_f,l_g} \cap P_{k_h,l_i} \times Q_{k_h,l_i} = \emptyset,$$

i.e., there is no pair of indices that is found in two different a-IMs in the given EIM A, where $\times$ is the standard Cartesian product.

In this case, if there are two or more p-indices in $A|^*$ that coincide, all members of the rows with these indices are written on the respective places in the first row with coinciding indices. Obviously, there are no two elements, that stay on one and the same place, because they are members of different IMs.

After this, if there are two or more q-indices in $A|^*$ that coincide, all members of the columns with these indices are written on the respective places in the first column with coinciding indices. Obviously, again, there are no two elements, that stay on one and the same place.

For example, for the IM A from the above example, if

$$a_{1,2} = \begin{array}{c|ccc}
 & q_{2,1} & q_{2,2} & q_{2,3} \\ \hline
p_{1,2} & c_{1,1} & c_{1,2} & c_{1,3} \\
p_{2,2} & c_{2,1} & c_{2,2} & c_{2,3}
\end{array},$$

$$a_{2,2} = \begin{array}{c|cc}
 & q_{3,1} & q_{4,2} \\ \hline
p_{4,1} & e_{1,1} & e_{1,2} \\
p_{4,2} & e_{2,1} & e_{2,2} \\
p_{4,3} & e_{3,1} & e_{3,2}
\end{array},$$

$$a_{3,1} = \begin{array}{c|c}
 & q_{2,1} \\ \hline
p_{3,1} & f_{1,1} \\
p_{3,2} & f_{2,1}
\end{array},$$

and

$$a_{3,2} = \begin{array}{c|cc}
 & q_{1,1} & q_{1,2} \\ \hline
p_{6,1} & g_{1,1} & g_{1,2}
\end{array},$$

then

$$A|^* = \begin{array}{c|cccccccc} & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,1} & q_{4,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,2} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{2,1} & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 \\ p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 \\ p_{3,1} & 0 & 0 & 0 & f_{1,1} & 0 & d_{1,1} & 0 & 0 \\ p_{4,1} & 0 & 0 & 0 & f_{2,1} & 0 & 0 & e_{1,1} & e_{1,2} \\ p_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} \\ p_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} \\ p_{6,1} & g_{1,1} & g_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

The following assertions are proved in [12].

Theorem 3. Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an EIM and let each its element a_{k_f,l_g} be an IM. Then

$$A|^* = \sum_{\substack{1 \le f \le m \\ 1 \le g \le n}} a_{k_f,l_g},$$

where symbol $\sum$ denotes the generalization of the operation addition $\oplus_{(\perp)}$. Here, the suboperation of the operation addition $\oplus_{(\perp)}$ is $\perp$, because it will be applied over no one pair of elements.

Theorem 4. Let A be an EIM and α be a fixed real (complex) number. Then

$$(\alpha A)|^* = \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \alpha a_{k_1,l_1} & \dots & \alpha a_{k_1,l_j} & \dots & \alpha a_{k_1,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & \alpha a_{k_i,l_1} & \dots & \alpha a_{k_i,l_j} & \dots & \alpha a_{k_i,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & \alpha a_{k_m,l_1} & \dots & \alpha a_{k_m,l_j} & \dots & \alpha a_{k_m,l_n} \end{array}$$

$$= \begin{array}{c|ccccc} & q_{k_1,l_1,1} & \dots & q_{k_1,l_1,s_{1,1}} & \dots & q_{k_m,l_n,s_{m,n}} \\ \hline p_{k_1,l_1,1} & \alpha a_{k_1,l_1,1,1} & \dots & \alpha a_{k_1,l_1,1,s_{1,1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{k_1,l_1,r_{1,1}} & \alpha a_{k_1,l_1,r_{1,1},1} & \dots & \alpha a_{k_1,l_1,r_{1,1},s_{1,1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{k_m,l_n,1} & 0 & \dots & 0 & \dots & \alpha a_{k_m,l_n,1,s_{m,n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{k_m,l_n,r_{m,n}} & 0 & \dots & 0 & \dots & \alpha a_{k_m,l_n,r_{m,n},s_{m,n}} \end{array}.$$

Now, we describe the case when A is an n-dimensional EIM with elements that are also EIMs.

Let

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, ..., k_{i,m_i}\}$, $m_i \ge 1$ and each one of its elements has the form

$$a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}$$

$$= [L^1_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, L^2_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, \ldots, L^{r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}}_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}},$$
$$\{b^{q_1,q_2,...,q_{r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}}}_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}],$$

where

- $\langle k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}\rangle \in K_1 \times K_2 \times \ldots \times K_n$,
- $r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \geq 1$ is the dimension of the EIM $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}$,
- for each i $n(1 \leq i \leq n)$ and for each j $(1 \leq j \leq r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}})$:

$$1 \leq q_j \leq |L^j_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}|$$

  for $|X|$ - the cardinality of set X and

$$K_i \cap L^j_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} = \emptyset, \tag{3}$$

- for each $j_1, j_2 (1 \leq j_1 < j_2 \leq r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}})$:

$$L^{j_1}_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \cap L^{j_2}_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} = \emptyset. \tag{4}$$

Then

$$A|^* = \left[\bigcup_{\langle k_{1,s_1},...,k_{n,s_n}\rangle \in K_1\times\ldots\times K_n} L^1_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, \ldots,\right.$$
$$\left.\bigcup_{\langle k_{1,s_1},...,k_{n,s_n}\rangle \in K_1\times\ldots\times K_n} L^n_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, \{c^{w_1,...,w_n}_{u_{1,v_1},...,u_{n,v_n}}\}\right],$$

where

$$c^{w_1,...,w_n}_{u_{1,v_1},...,u_{n,v_n}} = \begin{cases} b^{q_1,q_2,...,q_{r_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}}}_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, & \text{if } w_i = q_i, \text{ and } u_{i,v_i} = k_{i,s_i} \\ & \text{for } 1 \leq i \leq n \\ \perp, & \text{otherwise} \end{cases}.$$

We can formulate and prove

Theorem 5. Let

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

be an n-DEIM, let each its element $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}$ be an EIM and let the index sets of all a-elements satisfy (3) and (4). Then

$$A|^* = \sum_{\langle k_{1,s_1},...,k_{n,s_n}\rangle \in K_1\times\ldots\times K_n} a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}.$$

We can change condition (4) with a weaker one: for every two different n-tuples $\langle k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}\rangle$ and $\langle l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}\rangle$:

$$\prod_{i=1}^{n} L^i_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \bigcap \prod_{i=1}^{n} L^i_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} = \emptyset.$$

4 Aggregation operations over nDEIM

Let the nDEIM A be given and let $l_i \notin K_i$ for i $(1 \le i \le n)$ be n indices. First, by analogy with the paper of E. Sotirova, V. Bureva and the author [14], we introduce the following four aggregation operations over A:

Max-aggregation

$$\mathcal{A}_{(\max)}(A, i, l_i)$$

$$= \begin{array}{c|ccc} & k_{i,1} & \dots \\ \hline l_i & \max\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \max\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} \end{array},$$

Min-aggregation

$$\mathcal{A}_{(\min)}(A, i, l_i)$$

$$= \begin{array}{c|ccc} & k_{i,1} & \dots \\ \hline l_i & \min\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \min\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} \end{array},$$

Sum-aggregation

$$\mathcal{A}_{(\mathrm{sum})}(A, i, l_i)$$

$$= \begin{array}{c|ccc} & k_{i,1} & \dots \\ \hline l_i & \sum\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \sum\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} \end{array},$$

Average-aggregation

$$\mathcal{A}_{(\mathrm{ave})}(A, i, l_i)$$

$$= \begin{array}{c|ccc} & k_{i,1} & \dots \\ \hline l_i & \frac{1}{M_i}\left(\sum\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}}\right) & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \frac{1}{M_i}\left(\sum\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}}\right) \end{array},$$

where

$$M_i = \prod_{j=1, j\neq i}^{n} m_j.$$

We can see immediately that for every IM A, for every pair of indices i and j and for every $\circ \in \{\max, \min, \text{sum}, \text{ave}\}$:

$$\mathcal{A}_{(\circ)}(\mathcal{A}_{(\circ)}(A, j, l_j), i, l_i) = \mathcal{A}_{(\circ)}(\mathcal{A}_{(\circ)}(A, i, l_i), j, l_j).$$

In the case of $(0,1)$-nDEIMs, only operations $\mathcal{A}_{\max}$ and $\mathcal{A}_{\min}$ are possible.

When we use n-DEIM with elements propositions or predicates, then the aggregation operations are

$\vee$-**aggregation**

$$\mathcal{A}_{(\max)}(A, i, l_i)$$

$$= \begin{array}{c|cc} & k_{i,1} & \dots \\ \hline l_i & \bigvee\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \bigvee\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} \end{array},$$

and $\wedge$-**aggregation**

$$\mathcal{A}_{(\min)}(A, i, l_i)$$

$$= \begin{array}{c|cc} & k_{i,1} & \dots \\ \hline l_i & \bigwedge\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,1},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} & \dots \end{array}$$

$$\begin{array}{c|cc} & \dots & k_{i,m_i} \\ \hline l_i & \dots & \bigwedge\limits_{\langle k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i+1,s_{i+1}},\dots,k_{n,s_n}\rangle \in K_1\times\dots\times K_{i-1}\times K_{i+1}\times\dots\times K_n} a_{k_{1,s_1},\dots,k_{i-1,s_{i-1}},k_{i,m_i},k_{i+1,s_{i+1}},\dots,k_{n,s_n}} \end{array}.$$

5 n-DEIM with function-type of elements

The concept of a (2-dimensional) IM with function-type of elements was described in details in [6, 8, 10]. Here, we introduce its n-dimensional form.

Let the set of all used functions be $\mathcal{F}$.

The research over IMs with function-type of elements develops in the following two cases:

- each function of set $\mathcal{F}$ has one argument and it is exactly x (i.e., it is not possible that one of the functions has argument x and another function has argument y) – let us mark the set of these functions by $\mathcal{F}_x^1$;
- each function of set $\mathcal{F}$ has one argument, but that argument might be different for the different functions or the different functions of set $\mathcal{F}$ have different numbers of arguments.

The n-DEIM with Function-type of Elements (n-DEIMFE) has the form

$$A = [K_1, K_2, ..., K_n, \{f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, ..., k_{i,m_i}\}$, $m_i \geq 1$ and $f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{F}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$.

The second n-DEIM A representations is:

$$A = \left\{ \begin{array}{c|ccc} \langle k_{3,s_3}, ..., k_{n,s_n}\rangle & k_{2,1} & \dots & k_{2,m_2} \\ \hline k_{1,1} & f_{k_{1,1},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \dots & f_{k_{1,1},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,i} & f_{k_{1,i},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \dots & f_{k_{1,i},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,m_1} & f_{k_{1,m_1},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \dots & f_{k_{1,m_1},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \end{array} \right.$$

$$|\langle k_{3,s_3}, ..., k_{n,s_n}\rangle \in K_3 \times \ldots \times K_n\}.$$

The n-DEIMFE has this form independently of the form of its elements. They can be functions from $\mathcal{F}_x^1$ having one, exactly determined argument (in the present case - x), as well as functions with a lot of arguments. The set of s-argument functions will be marked by $\mathcal{F}^s$, where $s \geq 2$.

Now, we give the definitions of the operations over n-DEIMFEs. The first three of them are in more general form than in [6, 8].

Let the n-DEIMFEs

$$F = [K_1, K_2, ..., K_n, \{f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

and

$$G = [L_1, L_2, ..., L_n, \{g_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\}],$$

where $f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, g_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} \in \mathcal{F}_x^1$ or $f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}$, $g_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} \in \mathcal{F}^d$ simultaneously, where $d \geq 2$.

Following and extending [10], we introduce the operation "addition" over A and B by:

$$F \oplus_{(\circ)} G = [K_1 \cup L_1, K_2 \cup L_2, ..., K_n \cup L_n, \{h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}$$

$$= \begin{cases} f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ so that } k_{i,s_i} \notin L_i \\ \\ g_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ so that } l_{i,t_i} \notin K_i \\ \\ f_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i \\ \circ\, g_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} & \\ \\ \perp, & \text{otherwise} \end{cases},$$

where operation $\circ$ is defined in $\mathcal{F}_x^1$ or in $\mathcal{F}^d$, in respect of the form of the f- and g-functions. For example, if

$$f(x) = x^5 - 3x^4 + x^3 + 1,$$

$$g(x) = x^6 - 2x^5 - 3,$$

and operation $\circ$ is $+$, then

$$h(x) = f(x) \circ g(x) = x^6 - x^5 - 3x^4 + x^3 - 2.$$

Therefore, the operation addition over two nDEIMFEs is similar to this one from Section 1.3. The same is the situation with operations "termwise multiplication", "multiplication", "structural subtraction", "multiplication with a constant".

The next operations are specific for IMFEs (for any dimension).

Let

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

and

$$F = [L_1, L_2, ..., L_n, \{f_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\}],$$

be given, where $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{R}$ and $f_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} \in \mathcal{F}_x^1$ or $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{R}^d$ and $f_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} \in \mathcal{F}^d$ simultaneously, where $d \geq 2$. Then, we can define the following operations:

$$A \boxplus F = [K_1 \cup L_1, K_2 \cup L_2, ..., K_n \cup L_n, \{h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}$$

$$= \begin{cases} a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ so that } k_{i,s_i} \notin L_i \\ f_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ so that } l_{i,t_i} \notin K_i \\ a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i \\ \circ\, f_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}} & \\ \bot, & \text{otherwise} \end{cases},$$

where $\circ \in \{+, -, \times, :\}$ and $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{R}$.

$$F \boxplus A = [K_1 \cup L_1, K_2 \cup L_2, ..., K_n \cup L_n, \{h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},...,p_{n,q_n}}$$

$$= \begin{cases} a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ so that } k_{i,s_i} \notin L_i \\ f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ so that } l_{i,t_i} \notin K_i \\ f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}(& \\ a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}), & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i \\ \bot, & \text{otherwise} \end{cases},$$

where a- and f-elements belong of sets with equal dimensions.

$$A \boxtimes F = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} \circ f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}},$$

if for each i: $p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$, where $\circ \in \{+, -, \times, :\}$ and $a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} \in \mathcal{R}$.

$$F \boxtimes A = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}),$$

if for each i: $p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$, and a- and f-elements belong of sets with equal dimensions.

Finally, for the two n-DEIMFEs F and G, following [10] we define two operations "composition" by:

$$F \diamondsuit_{(\oplus)} G = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}$$

$$= \begin{cases} f_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ so that } k_{i,s_i} \notin L_i \\ g_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ so that } l_{i,t_i} \notin K_i \\ f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}(& \\ g_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}), & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i \\ \bot, & \text{otherwise} \end{cases},$$

and

$$F \diamondsuit_{(\otimes)} G = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = f_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}(g_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}),$$

where for each i: $p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$.

When the elements of the n-DEIMFE are propositions or predicates, the elements of set $\mathcal{F}^d$ will be logical functions.

6 Relations over n-DEIMs

Let the two n-DEIMs A and B be given. In [13], 6 relations over n-DEIMs with elements being real or natural numbers, or with elements of set $\{0, 1\}$ are introduced. There, the definitions where used for $\subset$ and $\subseteq$ that denote the relations *"strong inclusion"* and *"weak inclusion"*. They are:

The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \;\&\; (\exists j : 1 \le j \le n)(K_j \ne L_j))$$
$$\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} = b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } (\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$
$$= b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} < b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The non-strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} \le b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The non-strict relation "inclusion about value" is **The strict relation "inclusion"** is

$$A \subset_d B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \;\&\; (\exists j : 1 \le j \le n)(K_j \ne L_j))$$
$$\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} < b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The non-strict relation "inclusion" is

$$A \subseteq_d B \text{ iff } (\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$
$$\le b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

Let V be an evaluation function that estimates the truth-value of logical variables, propositions or predicates. Then, the first two above relations keep their form, but the four next relations change their forms, as follows.

The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \;\&\; (\exists j : 1 \le j \le n)(K_j \ne L_j))$$

$$\&(\forall i)(\forall j : 1 \leq j \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) = V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}})).$$

The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } (\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \leq j \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) = V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}})).$$

The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall s_i : 1 \leq s_i \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) < V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}))).$$

The non-strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall s_i : 1 \leq s_i \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) \leq V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}})).$$

The strict relation "inclusion" is

$$A \subset B \text{ iff } ((\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \ \& \ (\exists j : 1 \leq j \leq n)(K_j \neq L_j))$$
$$\&(\forall i)(\forall j : 1 \leq j \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) < V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}})).$$

The non-strict relation "inclusion" is

$$A \subseteq B \text{ iff } ((\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \leq j \leq m_i)(V(a_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}}) \leq V(b_{k_{1,s_1},k_{2,s_2},\ldots,k_{n,s_n}})).$$

7 Conclusion

The results from the present paper extend the idea for index matrices. In the future, we will introduce the same operations and relations for the case when the n-DEIMs have as elements IFPs.

Following the ideas from [14-18], we see that the n-DEIMs give a suitable model of the OLAP-cube.

In a next research, we will discuss n-DEIM with elements intuitionistic fuzzy pairs.

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